Algebraic c_2-invariants

or "how to compute Feynman amplitudes without doing any integration".

Motivation: G Feynman graph, amplitude I_G. Interested in the highest transcendental part of I_G.

Ex: $G =$  $I_G = 6\zeta(3)$

In QED (β-function) want to sum over graphs such as

Each one gives $a_i \zeta(3) + b_i$ (suitably regularized)

but the sum over all contributions is rational:

$$\sum a_i = 0 \qquad (*)$$

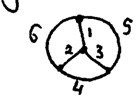
Cancellation problem: Observed: no $\zeta(3)$ at 3 loops.
$\zeta(3), \zeta(5)$ at 4 loops.

Goal: Find a combinatorial way to compute coeffs $a_i \in \mathbb{Q}$.
& try to prove $(*)$.

I). G connected graph, h_G loops, N edges.

Let $$\psi_G = \sum_{\substack{T \subseteq G \\ \text{s.t.}}} \prod_{e \notin T} \alpha_e \qquad \text{graph polynomial.}$$

T spanning trees.

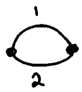
Eg:  s.t. 123 125 236 ..

$$\psi_G = \alpha_4\alpha_5\alpha_6 + \alpha_3\alpha_4\alpha_6 + \alpha_1\alpha_4\alpha_5 + \dots$$

Principal Feynman amplitude

$$I_G = \int_{\alpha_i \geq 0} \underbrace{\frac{\Omega_G}{\psi_G^2}}_{=:\omega_G^0} \qquad \Omega_G = \sum (-1)^i \alpha_i \, d\alpha_1 \dots \widehat{d\alpha_i} \dots d\alpha_N$$

Weinberg: converges if $\begin{cases} N = 2h_G \\ N_\gamma \geq 2h_\gamma + 1 \end{cases}$ $\forall \gamma \subsetneq G$.

Ex.: $G =$ 

$$I_G = \int_{\alpha_1, \alpha_2 \geq 0} \frac{\alpha_1 d\alpha_2 - \alpha_2 d\alpha_1}{(\alpha_1 + \alpha_2)^2} = \int_{\alpha_2 = 0}^{\infty} \frac{d\alpha}{(1+\alpha)^2} = 1.$$

Generalized Feynman integral

$$I = \int \underbrace{\frac{P}{\psi_G^M} \Omega_G}_{\omega} \qquad \begin{cases} M \in \mathbb{N} \\ P \in \mathbb{Q}[\alpha_1, \dots, \alpha_N] \\ \text{st. integrand homogeneous degree } 0. \end{cases}$$

It may or may not converge.

To every such $\underset{\omega}{\underline{\text{integrand}}}$; I want to associate rational number(s).

II). Cohomology

Toy graph motive $\qquad H^{N-1}_{dR}(\mathbb{P}^{N-1} \setminus X_G)$

$$X_G = V(\psi_G) \subseteq \mathbb{P}^{N-1} \qquad \{(\alpha_1 : \dots : \alpha_N),\ \psi_G(\alpha) = 0\}$$

graph hypersurface. $\mathbb{P}^{N-1} \setminus X_G$ is affine.

Let $\Omega^n(\mathbb{P}^{N-1} \setminus X_G; \mathbb{Q}) =$ global regular algebraic n-forms on $\mathbb{P}^{N-1} \setminus X_G$ which are defined over $\mathbb{Q}$.

On affine chart (eg $\alpha_N = 1$): $\displaystyle\sum_{\substack{M, I \\ |I| = n}} \frac{P_I}{\psi_G^M} d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_n}$

Differential $\qquad d : \Omega^n \longrightarrow \Omega^{n+1} \qquad d^2 = 0.$

$$H^n_{dR}(\mathbb{P}^{N-1} \setminus X_G) := H^n(\Omega^\bullet(\mathbb{P}^{N-1} \setminus X_G; \mathbb{Q})) \qquad \text{affine.}$$

We have $[\omega_G^0] \in H^{N-1}_{dR}(\mathbb{P}^{N-1} \setminus X_G) =: M_G$

Take another integrand $[\omega] \in$.

I want to compare $[\omega]$ & $[\omega_G^0]$.

There is an increasing filtration $W_\bullet$ such that

$$0 = W_{N-2} \subseteq W_{N-1} \subseteq \cdots \subseteq W_{2N-2} = M_G .$$

Thm (B, Doryn). If G prim. divergent and $N_G \geq 5$ then

$$W_{2N-6} = M_G .$$

Define:

$$gr_{max} M_G = M_G / W_{2N-7} M_G .$$

$$\boxed{M_G \longrightarrow gr_{max} M_G}$$

Thm (B, D). If G denominator reducible, then

$$gr_{max} M_G \cong \begin{cases} \mathbb{Q} & \text{spanned by } [\omega_G^0] \\ 0 & \text{if } G \text{ "weight drop"} \end{cases}$$

Previous: BEK wheels, Doryn Z_n, -

Restrict to former case.
All graphs with $h_G \leq 6$ & most graphs at 7, 8 loops are denom. red. & non w. drop.

Define:

$$c : \Omega^{N-1}(\mathbb{P}^{N-1} \setminus X_G) \longrightarrow \mathbb{Q}$$

$$\omega \longmapsto [\omega] / [\omega_G^0] \in gr_{max} M_G$$

Example: G as above.

Let
$$I_n = \int \underbrace{\left(\frac{\prod \alpha_i}{\Psi_G^2}\right)^n \frac{\Omega_G}{\Psi_G^2}}_{\omega_G^n} , \qquad n \geq 0$$

$G =$ 

$$c(\omega_G^n) = 1, \ \frac{1}{60}, \ \frac{1}{900}, \ \frac{47}{400400}, \ldots \qquad n = 0, 1, 2, \ldots$$

$$I_n = 6\zeta(3), \ \frac{6}{60}\zeta(3) - \frac{7}{360}, \ \frac{1}{900} \cdot 6\zeta(3) - \frac{173}{129600}, \ldots$$

$n = 5$: $\zeta(3) \sim 984571 / 819072 = 1.2020\,56(7)$ 6 digits.

III) Algorithm for c.

Suppose G denom. red. etc. Choose order on edge variables. Start with $\omega^{(0)} = \omega \in \Omega^{N-1}(\mathbb{P}^{N-1} \setminus X_G)$ and construct a sequence of forms by eliminating variables $\alpha_1, \dots, \alpha_N$

$$\underset{\substack{N-1 \\ \text{form}}}{\omega^{(0)}}, \underset{\substack{N-2 \\ \text{form}}}{\omega^{(1)}}, \dots, \underset{\substack{0 \\ \text{form}}}{\omega^{(N-1)}}$$

- Work on affine open $\alpha_N = 1$

$$\omega^{(i)} = F^{(i)} \, d\alpha_i \dots d\alpha_{N-1} \qquad F^{(i)} \text{ rational function.}$$

- Generic step. If $F^{(i)} = \dfrac{P}{f^a g^b}$ where $\begin{cases} f = f^i\alpha_i + f_i \\ g = g^i\alpha_i + g_i \end{cases}$

$$F^{(i+1)} = \operatorname{Res}\Big|_{\alpha_i = -\frac{f_i}{f^i}} F^{(i)} = \operatorname{Res}\Big|_{\alpha_i = -\frac{g_i}{g^i}} F^{(i)}$$

- Weight drop steps. If $F^{(i)} = \dfrac{P}{f}$ and $\operatorname{Res}\Big|_{\alpha_i = \frac{-f_i}{f^i}} F^{(i)} = 0$

then $\exists$ G st

$$\frac{\partial G}{\partial \alpha_i} = F^{(i)}. \qquad \text{Set} \qquad F^{(i+1)} = G\Big|_{\alpha_i = 0}.$$

- Remark: For denominator reducible graphs, $\exists$ ordering on edges such that $\omega^{(i)}$ is always one of these two types.

(V) Interpretation.

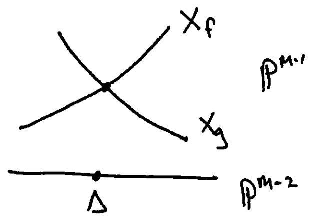
Second step: $f = f^i\alpha_i + f_i$ $\qquad X^i = V(f^i),\ X_i = V(f_i),\ X = V(f) \subseteq \mathbb{P}^M$

Need: $X^i \setminus (X^i \cap X_i)$ smooth $\qquad$ (B, Yeats, Schnetz)

$$\Rightarrow \quad H^{M-2}_{dR}\left(\mathbb{P}^{M-2} \setminus (X^i \cup X_i)\right) \longrightarrow H^{M-1}_{dR}\left(\mathbb{P}^{M-1} \setminus X\right)$$

is isom. on gr^W_{max} under certain assumptions.

First step:



Take residue along smooth locus of X_f and push downstairs. Idem for X_g & glue using Mayer-Vietoris.

IV Applications ?

a). For any gauge theory, Kreimer + al give parametric repⁿ.

Graphs $\rightsquigarrow$ Integrands $\sum \frac{N_i}{\Psi_G^M} \Omega_G$ ← underlying scalar topology

Apply c to integrands $\rightsquigarrow$ element in eg. Clifford algebra

Q1: Can we prove cancellation problem.

b). Q2: Kreimer & Broadhurst speculated on 4-term relations in gauge theories. Can it be proved "for the c-invariant"?

Point: c is defined for integrands irrespective of convergence of the integral.

c). Wißbrock et al. compute Mellin moments

$$\int \left(\frac{N}{\Psi_G}\right)^m \Omega_G \qquad \text{for all } m \geq 0$$

Satisfies P-F recurrence relation.
The c invariant is soln to this equation

d). Back to I_n, $G =$ 

$$I_n = a_n 6\zeta(3) + b_n$$

a_n, b_n solns to

$$u_n n^5 - 2(2n+1)(65n^4 + 130n^3 + 105n^2 + 40n + 6)\, u_{n+1}$$
$$+ \cancel{4}\,(4n+2)(4n+3)\cdots(4n+6)\, u_{n+2} = 0.$$

Apéry-type equation coming from physics.
Not good enough to prove $\zeta(3) \notin \mathbb{Q}$ ☹.

BUT  gives $a_n\zeta(5) + b_n\zeta(3) + c_n$ "very well poised".

Observation: There is no $\zeta(2)$!..... next lecture.